

A survey of the fixed point property in $\text{CAT}(\kappa)$ spaces

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Abstract. Since $\text{CAT}(\kappa)$ spaces are a generalization of a wide class of spaces, it is natural to ask how mappings defined on them behave. The main goal of this paper is to present the most important results on the fixed point property for two classes of mappings: nonexpansive and continuous ones. We consider both single- and set-valued mappings beginning with the basic theory due to Kirk from 2004 and finishing with the most recently obtained results devoted to the existence of fixed points for mappings defined on unbounded domains.

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1. Introduction

The concept of spaces with curvature bounded below or above (in the local sense) was introduced by Aleksandr Danilovich Alexandrov in the second half of the twentieth century. In eighties Mikhail Gromov introduced the notion of $\text{CAT}(\kappa)$ spaces (formed from the initial letters of Cartan – Alexandrov – Toponogov) for geodesic spaces with curvature bounded above by a real number (in the global sense):

- in case $\kappa \leq 0$ for any objects;
- in case $\kappa > 0$ for objects with perimeter not greater than $2\pi/\sqrt{\kappa}$.

Initially, the mathematicians' interest in $\text{CAT}(0)$ spaces focused on groups of isometries being the counterpart of Lie groups in differential geometry (compare eg. [3]). However, since $\text{CAT}(0)$ spaces are a very good generalization of some very well-known geometrical objects such as Hilbert spaces, metric trees, real and complex Hilbert balls with the hyperbolic metric etc..., it was quite natural to consider a much wider class

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of mappings defined on this kind of spaces and their fixed points. First positive results were obtained for nonexpansive mappings (defined on bounded and convex subsets of a CAT(0) space) by William A. Kirk in the turn of the century (compare e.g. [7, 12, 13]). Next some of these results were generalized for CAT(κ) spaces for any real κ (see [5]). Very recently authors have also begun to consider the fixed point property for continuous mappings defined on compact subsets of CAT(κ) spaces proving some counterparts of the Schauder Fixed Point Theorem and the Klee's result about geometrical conditions being equivalent to the fixed point property for continuous mappings (compare [2, 17]). At the same time another problem devoted to the existence of fixed points was considered. Namely, it is known that nonexpansive mappings defined on unbounded subsets of a geodesic space may behave in different ways depending on the curvature of a space. In case of Hilbert spaces the fixed point property cannot be guaranteed for any type of unbounded subsets (see [18]) while in the case of very particular examples of spaces with negative curvature this property is guaranteed if one only assumes that a domain is geodesically bounded (see [7, 9]). So it is natural to ask how such mappings behave in more general subclasses of CAT(0) spaces (compare [8, 21, 22]).

The purpose of this paper is to present the most important results devoted to the problem of existence of fixed points for nonexpansive and continuous mappings defined on a wide class of subsets of CAT(κ) spaces. The paper is organized as follows. In Section 2 we introduce the concept of the Model Space necessary to define CAT(κ) spaces. We also mention main properties of this type of spaces and define a very special case of CAT(0) spaces, namely, metric trees. In Section 3 we study the fixed point property for nonexpansive type mappings defined on bounded subsets of CAT(κ) spaces. We consider here two cases – single-valued and set-valued mappings. In the next section we consider the behaviour of mappings when the boundedness assumption is dropped. Finally, in Section 5 we focus on the fixed point property for continuous mappings for both single- and set-valued mappings.

2. Preliminaries

Let (X, ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map $c: [0, l] \subset \mathbb{R} \rightarrow X$ such that $c(0) = x$, $c(l) = y$, and $\rho(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $\rho(x, y) = l$. The image α of c is called a *metric segment* joining x and y . The metric space (X, ρ) is called *geodesic* if each pair of points of X can be joined by a metric segment.

Let (X, ρ) be a geodesic metric space. X is said to be uniquely geodesic if each pair of points of X is joined by a unique metric segment which will be denoted by $[x, y]$ for $x, y \in X$. A subset K of X is called convex if $[x, y] \subset K$ for every $x, y \in K$ and geodesically bounded if there is no so called unbounded geodesic in K . By an unbounded geodesic we mean the image of an isometry $c: \mathbb{R}_+ \rightarrow X$.

Now we introduce the concept of model spaces M_κ^2 , $\kappa \in \mathbb{R}$, which will be needed to define CAT(κ) spaces. In [3] the reader can find a very generous exposition on CAT(κ) spaces. First let us consider the space \mathbb{R}^3 endowed with the symmetric bilinear form

which associates to vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ the real number $\langle u|v \rangle$ defined by

$$\langle u|v \rangle = u_1v_1 + u_2v_2 - u_3v_3.$$

Let \mathbb{H}^2 be a set

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \langle x, x \rangle = -1 \wedge x_3 \geq 1\}.$$

Then \mathbb{H}^2 with a function $d: \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow \mathbb{R}$ defined by

$$d(u, v) = \text{arc cosh} \langle u, v \rangle$$

is a uniquely geodesic space. In a similar way one may define \mathbb{H}^n being a subset of \mathbb{R}^{n+1} , $n \in \mathbb{N}$, and an infinite dimensional space \mathbb{H}^∞ (the subset of the Hilbert space ℓ^2) being isometric to the real Hilbert ball with the hyperbolic metric (for a detailed exposition of the real and complex Hilbert ball see [9, Section II.32]).

Now let \mathbb{S}^2 be the unit sphere of \mathbb{R}^3 . For a pair of points $x, y \in \mathbb{S}^2$ we define a spherical distance $d(x, y)$ as

$$d(x, y) = \arccos(x|y),$$

where by $(x|y)$ we mean the scalar product in \mathbb{R}^3 , i.e.,

$$(x|y) = x_1y_1 + x_2y_2 + x_3y_3 \quad \text{for } x = (x_1, x_2, x_3) \text{ and } y = (y_1, y_2, y_3).$$

Then (\mathbb{S}^2, d) is a uniquely geodesic space as long as $d(x, y) < \pi$.

The Model Spaces M_κ^2 for $\kappa \in \mathbb{R}$ are defined as follows. Given $\kappa \in \mathbb{R}$, we denote by M_κ^2 the following metric space:

1. if $\kappa = 0$, then M_κ^2 is the Euclidean space \mathbb{E}^2 ;
2. if $\kappa < 0$, then M_κ^2 is obtained from the hyperbolic space \mathbb{H}^2 by multiplying the distance function by a constant $1/\sqrt{-\kappa}$;
3. $\kappa > 0$, then M_κ^2 is obtained from the spherical space \mathbb{S}^2 by multiplying the distance function by a constant $1/\sqrt{\kappa}$.

Now let us consider a geodesic space (X, ρ) . A *geodesic triangle* $\Delta(x, y, z)$ of X consists of three vertices $x_1, x_2, x_3 \in X$ and any metric segments $[x_i, x_j]$, $i, j \in \{1, 2, 3\}$. Now let us fix $\kappa \in \mathbb{R}$ and in the case of positive κ consider only triangles of perimeter smaller than $\frac{2\pi}{\sqrt{\kappa}}$. Then one can find a *comparison triangle* $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of M_κ^2 such that

$$\rho(x_i, x_j) = d(\bar{x}_i, \bar{x}_j), \quad i, j \in \{1, 2, 3\},$$

for the respective distance d of M_κ^2 . This triangle is unique up to isometries.

Definition 2.1. *The triangle $\Delta(x_1, x_2, x_3) \subset X$ is said to satisfy the $CAT(\kappa)$ inequality if for each pair of points $a, b \in \Delta(x_1, x_2, x_3)$ with $a \in [x_i, x_j]$, $b \in [x_i, x_k]$ and $\rho(x_i, a) = \alpha\rho(x_i, x_j)$, $\rho(x_i, b) = \beta\rho(x_i, x_k)$ and the comparison points $\bar{a}, \bar{b} \in \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, i.e., points for which $\bar{a} \in [\bar{x}_i, \bar{x}_j]$, $\bar{b} \in [\bar{x}_i, \bar{x}_k]$ with $d(\bar{x}_i, \bar{a}) = \alpha d(\bar{x}_i, \bar{x}_j)$ and $d(\bar{x}_i, \bar{b}) = \beta d(\bar{x}_i, \bar{x}_k)$ the following inequality*

$$\rho(a, b) \leq d(\bar{a}, \bar{b})$$

holds.

A metric space (X, ρ) is called a $\text{CAT}(\kappa)$ space if each triangle of X (for positive κ with perimeter $< 2\pi/\sqrt{\kappa}$) satisfies the $\text{CAT}(\kappa)$ inequality.

Now let us mention the basic properties of so defined spaces. First we quote a very well-known property of $\text{CAT}(\kappa)$ spaces which allows us to consider in the sequel only very particular cases of κ (see eg. [3]).

Proposition 2.2. *Let $\kappa < \kappa'$. Then each $\text{CAT}(\kappa)$ space is also a $\text{CAT}(\kappa')$ space. Moreover, if X is a $\text{CAT}(\kappa)$ space for all $\kappa > \kappa'$, then X is also a $\text{CAT}(\kappa')$ space.*

In the following result one may assume that X is a $\text{CAT}(\kappa)$ space with nonpositive κ , but on account of the previous proposition it suffices to consider only the case $\kappa = 0$.

Proposition 2.3. *Let X be a $\text{CAT}(0)$ space and C be a nonempty closed and convex subset of X . Then C is a retraction of X , i.e., there exists a nonexpansive mapping $R: X \rightarrow C$ such that $R|_C = \text{Id}$.*

Moreover, the retraction from the previous proposition may be chosen in the following way

$$R(x) = \left\{ \bar{c} \in C : \rho(x, \bar{c}) = \inf_{c \in C} \rho(c, x) \right\}.$$

Then we will say that R is a projection onto a subset C and denote this mapping by P_C .

We also have the following property:

Proposition 2.4. *Let C be a nonempty closed and convex subset of a $\text{CAT}(0)$ space. Then $P_C: X \rightarrow C$ satisfies*

$$\angle_{P_C(x)}(x, y) \geq \frac{\pi}{2}, \quad x \in X \text{ and } y \in C,$$

and $P_C(u) = P_C(x)$ for all $u \in [x, P_C(x)]$.

At the same time one can get the following property of projections in $\text{CAT}(\kappa)$ space if κ is a positive number:

Remark 2.5. Let us note that in the case of $\text{CAT}(\kappa)$ spaces with positive parameter κ the projection onto closed and convex subset may be defined under an additional assumption on the diameter of X but this mapping is still not nonexpansive. However, the projection is a lipschitzian mapping and the Lipschitz constant can be found in e.g. [20].

Now let us suppose that (x_n) is a bounded subset of a metric space X . Then we set

$$r(x, (x_n)) := \limsup_{n \rightarrow \infty} \rho(x_n, x).$$

We define the radius $r((x_n))$ of (x_n) by

$$r((x_n)) := \inf_{x \in X} r(x, (x_n))$$

and the asymptotic center $A((x_n))$ by

$$A((x_n)) := \{x \in X \mid r(x, (x_n)) = r((x_n))\}.$$

Let us note that depending on X the asymptotic center can be empty or can contain more than one point (here noncomplete spaces X can be considered, see for example [23]). But in case of a $CAT(\kappa)$ space we have:

Proposition 2.6. *Let X be a $CAT(\kappa)$ space and (x_n) be a bounded sequence of points of X (with $r((x_n)) < \pi/2\sqrt{\kappa}$ for $\kappa > 0$). Then the asymptotic center is a singleton.*

The proof of this result can be found in [5] and [15] (see also [6, Corollary 3.7]).

A very particular example of geodesic spaces are metric trees which have a lot of applications in different fields. This class of spaces can be treated as $CAT(\kappa)$ spaces for infinite parameter κ ($\kappa = -\infty$).

Definition 2.7. *A metric tree is a geodesic metric space M such that:*

- (1) *for all $x, y \in M$ there is unique metric segment $[x, y]$ joining them;*
- (2) *if x, y and $z \in M$ are such that $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.*

This definition implies some basic properties which are not shared with other subclasses of $CAT(0)$ spaces. We present them in the following proposition.

Proposition 2.8. *Let M be a metric tree. Then*

- (1) *for all $x, y, z \in M$ there is the unique $w \in M$ such that $[x, z] \cap [y, z] = [w, z]$;*
- (2) *if C is a closed and convex subset of M , then for all $x \in C$ and all $y \in M$ we have*

$$\rho(x, y) = \rho(x, P_C(y)) + \rho(P_C(y), y),$$

so the projection $P_C(y)$ is a gate (see [1]).

3. Fixed points on bounded sets

We begin with a basic fixed point theorem for nonexpansive mappings defined on nonempty closed convex and bounded subsets of $CAT(\kappa)$ spaces. Since this theorem will be a fundamental result for the rest of this section we add a short proof. It is worth to emphasize that the same type of proof also works for uniformly convex Banach spaces and, more general, for uniformly convex metric spaces with a modulus of convexity that is monotone or lower-semicontinuous from the right.

First we consider subsets of $CAT(0)$ spaces.

Theorem 3.1. *Let C be a nonempty convex closed and bounded subset of a complete $CAT(0)$ space X and $T: C \rightarrow C$ be a nonexpansive mapping. Then the set of fixed points of T (denoted by $Fix T$) is nonempty closed and convex.*

Proof. Let us fix x_0 and consider an orbit $\{T^n(x_0) : n \in \mathbb{N}\}$, where $T^1(x) := T(x)$ and $T^{n+1}(x) := T(T^n(x))$ for all $x \in C$. This orbit forms a bounded sequence since C is bounded, so its asymptotic center is a singleton. Let us denote $A := A((T^n(x_0))) \in X$. It is easy to see that it must be that $A \in C$. Otherwise, considering the projection of A onto C the nonexpansivity of the projection mapping implies that

$$\rho(T^n(x_0), P_C(A)) = \rho(P_C(T^n(x_0)), P_C(A)) \leq \rho(T^n(x_0), A)$$

and $r(P_C(A), (x_n)) \leq r(A, (x_n))$ which contradicts the fact that A is an asymptotic center of $T^n(x_0)$.

Now repeating our consideration for $T(A)$ instead of $P_C(A)$ the uniqueness of asymptotic center implies that $A = T(A)$, i.e., the set of fixed points is nonempty. The closedness follows immediately from the fact that being nonexpansive the mapping T must be continuous. Finally, we want to show that $Fix T$ is convex. Let $x, y \in Fix T$ and $z \in [x, y]$. Then

$$\rho(x, T(z)) = \rho(T(x), T(z)) \leq \rho(x, z) \quad \text{and} \quad \rho(y, T(z)) = \rho(T(y), T(z)) \leq \rho(y, z),$$

and from the uniqueness of metric segments it follows that $T(z) \in [x, y]$. Moreover, if one assume that $\rho(x, z) = \alpha\rho(x, y)$, then $\rho(x, T(z)) \leq \alpha\rho(x, y)$ and $\rho(y, T(z)) \leq (1 - \alpha)\rho(x, y)$, which completes the proof that $T(z) = z$. \square

In a similar way one may obtain the same result for asymptotic nonexpansive and asymptotic pointwise nonexpansive mappings, i.e., the mappings for which

$$\rho(T^n(x), T^n(y)) \leq k_n\rho(x, y) \quad \text{and} \quad \limsup k_n \leq 1$$

or

$$\rho(T^n(x), T^n(y)) \leq k_n(x)\rho(x, y) \quad \text{and} \quad \limsup k_n(x) \leq 1 \quad \text{for each } x \in C$$

holds, respectively (cf. [11, Theorem 5.1] or [6, Theorem 3.11]).

Now let us consider a more general situation in which $T(C)$ does not have to be a subset of C .

Theorem 3.2. *Let C be a nonempty convex closed and bounded subset of a complete $CAT(0)$ space X and $T: C \rightarrow X$ be a nonexpansive mapping. Then there is a best proximity point, i.e., a point $x_0 \in C$ such that*

$$\rho(x_0, T(x_0)) = \inf_{x \in C} \rho(x, T(x_0)).$$

Remark 3.3.

1. Let us note that the point x_0 from the previous theorem is the unique projection of $T(x_0)$ onto C . Clearly, we do not claim that x_0 is the unique point satisfying the above condition.
2. Since the domain C of T is a closed convex subset of a complete $CAT(0)$ space X , C is a retract of the whole space X and the projection map $P_C: X \rightarrow C$ is nonexpansive (see Proposition 2.3), so one may define a new mapping $\tilde{T} = P_C \circ T$ which, as a composition of nonexpansive mappings, is also nonexpansive. On account of Theorem 3.1 there is a fixed point x_0 of \tilde{T} . Hence $T(x_0) = x_0$ or x_0 is a projection of $T(x_0)$ onto C . In both cases the claim of the theorem holds.

Theorem 3.1 may be generalized for subsets of a $CAT(\kappa)$ space X for all $\kappa \in \mathbb{R}$. In case of negative κ , in the virtue of Proposition 2.2, one may repeat the above proof. If $\kappa > 0$, then we may obtain the following result assuming a boundedness condition on the radius of C (cf. [5, Theorem 3.9]):

Theorem 3.4. *Let C be a nonempty convex and closed subset of a complete $CAT(\kappa)$ space X , $\kappa > 0$, and $T: C \rightarrow C$ be a nonexpansive mapping. If the radius $rad_C(C) = \inf_{x \in C} \{ \sup_{y \in C} \rho(x, y) \}$ of C is smaller than $\frac{\pi}{2\sqrt{\kappa}}$, then the set of fixed points of T is nonempty closed and convex.*

The proof follows the same patterns as in Theorem 3.1.

We have repeated the result for positive κ to emphasize differences in behaviour between mappings $T: C \rightarrow X$ defined on subsets of $CAT(0)$ spaces and $CAT(\kappa)$ spaces for $\kappa > 0$. In both cases if C is bounded (for $\kappa > 0$ additionally $rad_C(C)$ cannot be too large) and $T: C \rightarrow C$, then there is a fixed point of T , i.e., a point x_0 which is the best proximity one, because $\rho(x_0, T(x_0)) = 0 = \inf_{x \in C} \rho(x, T(x_0))$. Now let us focus on mappings for which $T(C)$ is not contained in C . Theorem 3.1 and the properties of the projection mapping guarantee that T has a best proximity point for $\kappa \leq 0$. In the case of positive κ , the situation may be completely different as the following example shows.

Example 3.5. Let us consider the unit sphere \mathbb{S}^2 . Clearly, we may view it as a $CAT(1)$ space. Take C a closed ball of \mathbb{S}^2 centered at the north pole and with radius less than $\frac{\pi}{2}$. Let $T: C \rightarrow \mathbb{S}^2$ be defined by $T(x) = -x$, i.e., for $x = (x_1, x_2, x_3)$ we have $T(x) = (-x_1, -x_2, -x_3)$. Obviously, being an isometry T is also a nonexpansive mapping. Now let us consider any point of $T(C)$. If $y \in T(C)$ satisfies $T^{-1}(y) = P_C(y)$, then y must belong to the boundary of C . In that case

$$P_C(y) = (y_1, y_2, -y_3), \quad \text{for } y = (y_1, y_2, y_3)$$

while

$$T^{-1}(y) = (-y_1, -y_2, -y_3) \neq P_C(y).$$

So there is no best proximity point in C .

In [10] the fixed point property for set-valued mappings has been considered. Before we pass to this result we have to recall some definitions. Let a set-valued mapping $T: X \rightarrow 2^X$ take as values nonempty subsets of X . In this case x will be called a fixed point of F if and only if $x \in F(x)$. Obviously, this is not the unique way to define fixed points for set-valued mappings, but we use this definition in the sequel. In case of a metric space (X, ρ) first we define an ε -hull of a nonempty set $C \subset X$. Namely, $N_\varepsilon(C)$ is defined by

$$N_\varepsilon(C) = \{x \in X \mid \rho(x, C) = \inf_{c \in C} \rho(x, c) < \varepsilon\}.$$

Next we consider the family of nonempty and closed subsets of X which we denote by $cl(X)$. Then $(cl(X), h)$ is a metric space, where

$$h(C_1, C_2) = \inf_{\varepsilon > 0} \{C_i \subset N_\varepsilon(C_j), i, j \in \{1, 2\}\}.$$

If additionally X is complete, then so is $cl(X)$. Moreover, instead of closed subsets one can take only compact subsets $cc(X)$ and then the metric space $(cc(X), h)$ is still complete (cf. [4, Theorems II-2, II-3 and II-5, pp. 38–41]). In this case we say that $T: X \rightarrow cc(X)$ is a nonexpansive mapping if

$$h(T(x), T(y)) \leq \rho(x, y), \quad x, y \in X.$$

The following counterpart of Theorem 15.3 from [10] holds in the setting of CAT(0) spaces (cf. [6, Theorem 4.2]):

Theorem 3.6. *Let X be a complete CAT(0) space and C a nonempty closed convex and bounded subset of X . Then each nonexpansive set-valued mapping $T: C \rightarrow cc(C)$ has at least one fixed point, i.e., there exists a point $x \in C$ such that $x \in T(x)$.*

Under an additional assumption on the radius of C the same result is true for complete CAT(κ) spaces with a positive κ (see [6, Theorem 4.3]).

4. Fixed points on unbounded sets

In 1980 William O'Ray proved that a closed and convex subset C of a Hilbert space has the following property:

Theorem 4.1 (O'Ray, 1980). *Let C be defined as above. Then C has the fixed point property for nonexpansive mappings if and only if C is bounded.*

Four year later the counterpart of this result was proved by Kazimierz Goebel and Simeon Reich for spaces of constant negative curvature:

Theorem 4.2 (Goebel, Reich, 1984). *Let X be a real Hilbert ball with the hyperbolic metric and $C \subset X$ a closed and convex subset. Then C has the fixed point property for nonexpansive mappings if and only if C is geodesically bounded.*

Since the real Hilbert ball with the hyperbolic metric is a space with constant curvature equal to -1 , we can extend this result for all separable spaces of constant negative curvature. Moreover, one can get the same equivalence for complete metric trees, which can be treated as spaces of curvature equal to $-\infty$ (cf. [13]). So the natural question one may raise is whether there are any geometric conditions which are equivalent to the fixed point property in CAT(0) spaces. This problem was considered for the first time by Rafa Espínola and the author in [8]. In the same paper the reader can find many examples of subclasses of CAT(0) spaces for which the counterpart of Theorem 4.2 holds. Moreover, very recently the author gave in [21] the final answer for the question on the behaviour of spaces with strictly negative curvature.

Theorem 4.3 (Piątek, 2014). *Let X be a complete CAT(κ) space, $\kappa < 0$ and $C \subset X$ a closed and convex subset of X . Then C has the fixed point property for nonexpansive mappings if and only if C is geodesically bounded.*

5. Continuous mappings

In the previous section we mentioned the result due to William A. Kirk on metric trees. But we considered it only from the viewpoint of a very particular class of nonexpansive mappings. However, this result can be extended in the following way for all continuous mappings.

Theorem 5.1 (Kirk, 2004). *Let X be a complete metric tree. Then each continuous mapping $F: X \rightarrow X$ has at least one fixed point if and only if X is geodesically bounded.*

Next this result was explored for a wide class of continuous set-valued mappings.

Definition 5.2. *Let X be a metric tree and $F: X \rightarrow 2^X$ take nonempty values. Then we say that F is upper-semicontinuous at $x_0 \in X$ if for each open $V \subset X$ such that $F(x_0) \subset V$ there is an open $U \ni x_0$ with*

$$F(x) \subset V, \quad x \in U.$$

We say that F is lower-semicontinuous at $x_0 \in X$ if for each open $V \subset X$ such that $F(x_0) \cap V \neq \emptyset$ there is an open $U \ni x_0$ with

$$F(x) \cap V \neq \emptyset, \quad x \in U.$$

Clearly, these definitions can be used for a much wider class of spaces not only for metric trees. But in this particular case we obtain the following result (cf. [14] and [16]):

Theorem 5.3. *Let X be a complete and geodesically bounded metric tree. Then each upper- (lower-) semicontinuous set-valued mapping $F: X \rightarrow 2^X$ with nonempty closed convex and bounded values has at least one fixed point.*

Next the author introduced in [19] a weaker notion of semi-continuity. In case of metric spaces both upper and lower semi-continuous mappings belong to this new class.

Definition 5.4. *Let X be a metric tree and $F: X \rightarrow 2^X$ take nonempty values. Then we say that F is ε -semicontinuous at $x_0 \in X$ if for each positive $\varepsilon > 0$ there is an open $U \ni x_0$ such that*

$$F(x) \cap N_\varepsilon(F(x_0)) \neq \emptyset, \quad x \in U.$$

For these mappings it can be also shown that:

Theorem 5.5 (Piątek, 2008). *Let X be a complete and geodesically bounded metric tree. Then each ε -semicontinuous set-valued mapping $F: X \rightarrow 2^X$ with nonempty closed convex and bounded values has at least one fixed point.*

Clearly, one cannot expect that the similar result is still true in each complete $\text{CAT}(0)$ space. However, in this setting the counterpart of the Schauder Fixed Point Theorem holds (cf. [2, Corollary 18], also [17]):

Theorem 5.6 (Ariza-Ruiz, Li, López-Acedo, 2014). *Let X be a $\text{CAT}(0)$ space and K a nonempty closed convex and bounded subset of X . Then, any continuous $T: K \rightarrow K$ with compact range $T(K)$ has at least one fixed point in K .*

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